STRESS AND RATE OF STRAIN DISCONTINUITIES IN THE THREE-DIMENSIONAL CASE OF A COMPRESSIBLE RIGID-PLASTIC BODY

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Strong discontinuities in the stresses and the rates of strain in a compressible rigid-plastic body are investigate.

Changes in the relationships at the stress discontinuity surface [1] are obtained for an arbitrary condition of plasticity, the latter depending on the first invariant of the stress tensor.

The generalized condition of the Huber-von Mises [2] limit equilibrium is investigated as an illustration of the use of the relations obtained.

The restrictions imposed on the components of the rate of strain tensor are determined.

1. Let us consider a rigid-plastic solid whose limiting state is described by

$$f(\sigma, \Sigma_2, \Sigma_3) = 0 \qquad (\sigma = {}^1/_3\sigma_{ii}) \tag{1.1}$$

Here σ is the first invariant of the stress tensor, while Σ_2, Σ_3 are the second and third invariant of the stress deviator, respectively.

The associated law of plastic flow is

$$\varepsilon_{ij} = {}^1/_2(u_{i,j} + u_{j,i}) = \lambda\partial f / \partial\sigma_{ij} \qquad (\lambda \geqslant 0) \tag{1.2}$$

Here λ is an undetermined multiplier and $u_{i,j}$ is the partial derivative of the projection of the rate of displacement on the x_j-axis. From (1.2) follows [3]

$$\varepsilon_{ii} = \lambda\partial f / \partial\sigma \tag{1.3}$$

which defines the "associated" compressibility of the material.

Assume that a surface S exists in an isotropic rigid-plastic body, which is the stress discontinuity surface. The stresses which are in contact at the surface S must be continuous. Therefore, from the conditions of equilibrium it follows that

$$[\sigma_{ij}]\nu_j = 0 \qquad ([\sigma_{ij}] = \sigma_{ij}^+ - \sigma_{ij}^-) \tag{1.4}$$

Here the upper plus and minus signs denote the stresses at each side of S and ν_j is the unit normal to this surface.

The state of stress at each side of S must satisfy the limiting state condition (1.1). It follows therefore that

$$[f(\sigma, \Sigma_2, \Sigma_3)] = f(\sigma^+, \Sigma_2^+, \Sigma_3^+) - f(\sigma^-, \Sigma_2^-, \Sigma_3^-) = 0 \tag{1.5}$$

Let us assume that the components of the tensor ε_{ij} become discontinuous on passing through S. Then the geometrical compatibility conditions at the surface of discontinuity become [1]

$$[\varepsilon_{ij}] = {}^1/_2(\lambda_i\nu_j + \lambda_j\nu_i) = [\lambda P_{ij}], \quad \lambda_i = [u_{i,j}]\nu_j, \quad P_{ij} = \partial f / \partial\sigma_{ij} \tag{1.6}$$

Following [4], we shall show that for the convex conditions of plastic flow at the stress discontinuity surface in a compressible body the plastic deformation rates become equal to zero. To do this we define a local coordinate system at a point on S such, that the normal ν_j to S coincides with the direction of the x_3 -axis. Then

$$\nu_1 = \nu_2 = 0, \qquad \nu_3 = 1 \tag{1.7}$$

In the local coordinate system (1.7) the relations (1.4) and (1.6) give, respectively,

$$[\sigma_{13}] = [\sigma_{23}] = [\sigma_{33}] = 0, \qquad [\varepsilon_{11}] = [\varepsilon_{12}] = [\varepsilon_{22}] = 0 \tag{1.8}$$

These in turn imply that

$$[\sigma_{ij}]\,[\varepsilon_{ij}] = 0 \tag{1.9}$$

holds on S On the other hand, for the convex plastic flow surfaces the principle of the maximum rate of dissipation of mechanical energy implies that

$$[\sigma_{ij}]\,[\varepsilon_{ij}] > 0 \tag{1.10}$$

Comparing (1.9) and (1.10) we conclude that the relations

$$\varepsilon_{ij}^{+} = \varepsilon_{ij}^{-} = 0 \tag{1.11}$$

hold at the stress discontinuity surface.

In [1] it was shown that in the latter case $[u_{i,j}] = 0$ and the following relations hold on S

$$[a_{ij}] = {}^1\!/_2\,(c_i\nu_j + c_j\nu_i) = [\psi P_{ij}]$$
$$a_{ij} = \varepsilon_{ij,k\ldots l}\nu_k\ldots\nu_l, \quad c_i = [u_{i,jk\ldots l}]\,\nu_j\nu_k\ldots\nu_l, \quad \psi = \lambda_{,k\ldots l}\nu_k\ldots\nu_l \tag{1.12}$$

To find c_i we multiply (1.12) by ν_j and sum over the repeated indices. This gives

$$c_i = 2\,[\psi P_{ij}]\,\nu_j - [\psi P_{kk}]\,\nu_i, \qquad [\psi P_{kk}] = [a_{kk}] = c_k\nu_k \tag{1.13}$$

Since the matrial is assumed to be plastically compressible, the quantity $[\psi P_{kk}]$ should not vanish at the surface of discontinuity S. On inserting c_i given by (1.13) into (1.12) the latter become

$$[\psi P_{ik}]\,\nu_k\nu_j + [\psi P_{jk}]\,\nu_k\nu_i - [\psi P_{kk}]\,\nu_i\nu_j = [\psi P_{ij}] \tag{1.14}$$

only three of which are linearly independent. Relations (1.14) together with (1.4) and (1.5) form a closed system of seven equations defining the unknowns σ_{ij} and ψ^-. In the local coordinate system (1.7) the system (1.4)(1.5) and (1.14) becomes

$$[f(\sigma, \Sigma_2, \Sigma_3)] = 0, \quad [\sigma_{i3}] = 0, \quad [\psi P_{11}] = [\psi P_{12}] = [\psi P_{22}] = 0 \tag{1.15}$$

2. Let us investigate the relations (1.14) applied to the generalized Huber-von Mises limiting equilibrium condition, well-known in the soil mechanics.

$$S_{ij}S_{ij} = 2\,(\beta - \alpha\sigma)^2 \qquad (S_{ij} = \sigma_{ij} - {}^1\!/_3\sigma_{kk}\delta_{ij}, \quad \sigma \leqslant \beta/\alpha) \tag{2.1}$$

Here S_{ij} is the stress deviator and δ_{ij} is the Kronecker delta, while α and β are physical constants.

In this case the relations (1.15) become

$$[S_{ij}S_{ij} - 2\,(\beta - \alpha\sigma)^2] = 0 \tag{2.2}$$

$$[\psi\,(3S_{11} + 2\alpha\beta - 2\alpha^2\sigma)] = 0, [\psi\,(3S_{22} + 2\alpha\beta - 2\alpha^2\sigma)] = 0, \quad [\psi S_{12}] = 0 \tag{2.3}$$

$$[\sigma_{i3}] = 0 \tag{2.4}$$

Using the relations (2.4), together with the property $S_{kk} = 0$ valid for the diagonal components of the stress daviator, we can write (2.2) in the form

$$[S_{11}^2 + S_{11}S_{22} + S_{22}^2 + S_{12}^2 + 2\alpha\beta\sigma - \alpha^2\sigma^2] = 0 \qquad (2.5)$$

Inserting the values of the components of σ_{ij}^- obtained from (2.3) into (2.5) we obtain, after some manipulations,

$$(\psi^- / \psi^+)^2 = 1 \qquad (2.6)$$

From (2.3) it follows that the stress discontinuity is only possible when $\psi^- / \psi^+ = -1$, and when the following relations hold at S :

$$\sigma_{11}^- = 2\sigma_{33}^+\gamma_1 - \gamma_2 - \sigma_{11}^+, \qquad \sigma_{22}^- = 2\sigma_{33}^+\gamma_1 - \gamma_2 - \sigma_{22}^+ \qquad (2.7)$$

$$\sigma_{33}^- = \sigma_{33}^+, \qquad \sigma_{12}^- = -\sigma_{12}^+, \qquad \sigma_{13}^- = \sigma_{13}^+, \qquad \sigma_{23}^- = \sigma_{23}^+$$

$$\gamma_1 = (3 + 2\alpha^2) / (3 - 4\alpha^2), \qquad \gamma_2 = 12\alpha\beta / (3 - 4\alpha^2) \qquad (2.8)$$

The components of the stress tensor σ_{ij} can be expressed in terms of three principal stresses σ_j using the following transformation

$$\sigma_{ij} = \sigma_1 l_i l_j + \sigma_2 m_i m_j + \sigma_3 n_i n_j \qquad (2.9)$$

Here l_i, m_i and n_i are the direction cosines of the principal axes, connected by the following relations:

$$l_i l_j + m_i m_j + n_i n_j = \delta_{ij} \qquad (2.10)$$

Inserting (2.9) into (2.7) and taking into account (2.10) we obtain a system of twelve equations in σ_1^-, σ_2^-, σ_3^-, l_i^-, m_i^- and n_i^-. Solution of this system can be written in the following form

$$\sigma_1^- = 2\sigma_{33}\gamma_1 - \gamma_2 - \sigma_1^+, \qquad \sigma_2^- = 2\sigma_{33}\gamma_1 - \gamma_2 - \sigma_2^+, \qquad \sigma_3^- = 2\sigma_{33}\gamma_1 - \gamma_2 - \sigma_3^+$$

$$l_1^- = \pm l_1^+, \qquad m_1^- = \pm m_1^+, \qquad n_1^- = \mp n_1^+$$

$$l_2^- = \pm l_2^+, \qquad m_2^- = \pm m_2^+, \qquad n_2^- = \mp n_2^+ \qquad (2.11)$$

$$l_3^- = \mp l_3^+, \qquad m_3^- = \mp m_3^+, \qquad n_3^- = \pm n_3^-$$

Thus from [1] and (2.11) it follows that the plastic compressibility of the material affects only the magnitude of the stresses. The corresponding principal axes remain equally inclined to the surface S and lie on the planes passing through the normal to S.

3. We assume that the rates of strain ε_{ij} become discontinuous at some surface Σ

We have shown previously that under the convex conditions of plastic flow the rates of deformation become zero at the stress discontinuity surface in a compressible rigid-plastic body, i.e. they are continuous.

From this it follows that the stress discontinuity surface S and the rate of strain discontinuity surface Σ do not coincide.

Let us assume that the rates of displacement are continuous on Σ and find the restrictions which must be imposed on the discontinuities in the values of the components of ε_{ij} for solutions of the system (1.6) to exist.

Multiplying (1.6) by ν_j we find

$$\lambda_i = 2\,[\varepsilon_{ij}]\,\nu_j - [\varepsilon_{kk}]\,\nu_i \qquad (3.1)$$

where ν_i is the unit normal to Σ.

The condition of plastic compressibility implies that the sum $[\varepsilon_{kk}]$ does not vanish.

Then the system (1.6) can be written in the form

$$[\varepsilon_{ij}] = [\varepsilon_{ik}]\nu_k\nu_j + [\varepsilon_{jk}]\nu_k\nu_i - [\varepsilon_{kk}]\nu_i\nu_j \tag{3.2}$$

In the local coordinate system (1.7) the relations (3.2) become

$$[\varepsilon_{11}] = [\varepsilon_{12}] = [\varepsilon_{22}] = 0 \tag{3.3}$$

The components ε_{13}, ε_{23} and ε_{33} undergo a jump on the surface Σ. Choosing the axes x_i to coincide with the principal axes of the rate of deformation tensor, we obtain from (3.2)

$$\begin{aligned}
&[\varepsilon_{12}] = -[\varepsilon_3]\nu_1\nu_2 = 0, \quad [\varepsilon_1](1-\nu_1^2) + [\varepsilon_2]\nu_1^2 + [\varepsilon_3]\nu_1^2 = 0\\
&[\varepsilon_{13}] = -[\varepsilon_2]\nu_1\nu_3 = 0, \quad [\varepsilon_1]\nu_2^2 + [\varepsilon_2](1-\nu_2^2) + [\varepsilon_3]\nu_2^2 = 0\\
&[\varepsilon_{23}] = -[\varepsilon_1]\nu_2\nu_3 = 0, \quad [\varepsilon_1]\nu_3^2 + [\varepsilon_2]\nu_3^2 + [\varepsilon_3](1-\nu_3^2) = 0
\end{aligned} \tag{3.4}$$

The condition of compressibility and the existence of discontinuities on the surface Σ imply that the determinant of the three right hand side equations of (3.4) must be equal to zero, i.e.

$$\nu_1^2\nu_2^2\nu_3^2 = 0 \tag{3.5}$$

This in turn implies that either one or two of the principal axes lie in a plane tangent to Σ. Relations (3.4) and (3.5) yield various solutions for the discontinuities in the values of the components of the rate of deformation tensor

$$\begin{aligned}
&\nu_1 = 0, \quad \nu_2 \neq 0, \quad \nu_3 \neq 0, \quad [\varepsilon_1] = 0, \quad [\varepsilon_2] = (1-\nu_3^{-2})[\varepsilon_3]\\
&\nu_1 \neq 0, \quad \nu_2 = 0, \quad \nu_3 \neq 0, \quad [\varepsilon_2] = 0, \quad [\varepsilon_1] = (1-\nu_3^{-2})[\varepsilon_3]\\
&\nu_1 \neq 0, \quad \nu_2 \neq 0, \quad \nu_3 = 0, \quad [\varepsilon_3] = 0, \quad [\varepsilon_1] = (1-\nu_2^{-2})[\varepsilon_2]\\
&\nu_1 = \nu_2 = 0, \quad \nu_3 = 1, \quad [\varepsilon_1] = [\varepsilon_2] = 0, \quad [\varepsilon_3] \neq 0\\
&\nu_1 = \nu_3 = 0, \quad \nu_2 = 1, \quad [\varepsilon_1] = [\varepsilon_3] = 0, \quad [\varepsilon_2] \neq 0\\
&\nu_2 = \nu_3 = 0, \quad \nu_1 = 1, \quad [\varepsilon_2] = [\varepsilon_3] = 0, \quad [\varepsilon_1] \neq 0
\end{aligned} \tag{3.6}$$

Thus the above solutions impose restrictions on the magnitude of the discontinuities in the components of the rate of deformation tensor, such as are necessary for the system (1.6) to have a solution.

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